The uniformly frustrated two-dimensional XY model in the limit of weak frustration

Vincenzo Alba[1]**, Andrea Pelissetto**[2] **and Ettore Vicari**[3]

[1] Scuola Normale Superiore and INFN, I-56126 Pisa, Italy
[2] Dipartimento di Fisica dell'Università di Roma 'La Sapienza' and INFN, Sezione di Roma I, I-00185 Roma, Italy
[3] Dipartimento di Fisica dell'Università di Pisa and INFN, Sezione di Pisa, I-56127 Pisa, Italy

E-mail: Andrea.Pelissetto@roma1.infn.it and Ettore.Vicari@df.unipi.it

Abstract

We consider the two-dimensional uniformly frustrated XY model in the limit of small frustration, which is equivalent to an XY system, for instance a Josephson junction array in a weak uniform magnetic field applied along a direction orthogonal to the lattice. We show that the uniform frustration (equivalently, the magnetic field) destabilizes the line of fixed points which characterize the critical behavior of the XY model for $T \leqslant T_{\mathrm{KT}}$, where T_{KT} is the Kosterlitz–Thouless transition temperature: the system is paramagnetic at any temperature for sufficiently small frustration. We predict the critical behavior of the correlation length and of gauge-invariant magnetic susceptibilities as the frustration goes to zero. These predictions are fully confirmed by the numerical simulations.

PACS numbers: 74.81.Fa, 64.60.Fr, 05.10.Ln

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The uniformly frustrated two-dimensional (2D) XY model is defined by the lattice Hamiltonian

$$\mathcal{H} = -\sum_{\langle xy \rangle} \mathrm{Re}\, \psi_x U_{xy} \psi_y^* = -\sum_{\langle xy \rangle} \cos(\theta_x - \theta_y + A_{xy}), \qquad (1)$$

where $\psi_x \equiv \mathrm{e}^{\mathrm{i}\theta_x}$ and $U_{xy} \equiv \mathrm{e}^{\mathrm{i}A_{xy}}$. 2D arrays of coupled Josephson junctions in a magnetic field are interesting physical realizations of this model [1]. In this case, the sum $C(P_{nm})$ of the variables A_{xy} along the links of an elementary plaquette P_{nm},

$$C(P_{nm}) \equiv A_{(n,m),(n+1,m)} + A_{(n+1,m),(n+1,m+1)} - A_{(n,m+1),(n+1,m+1)} - A_{(n,m),(n,m+1)}, \qquad (2)$$

is related to the flux of an external magnetic field applied along an orthogonal direction: $C(P_{nm}) = a^2 B/\Phi_0$, where a is the lattice spacing, B is the magnetic field and $2\Phi_0 = hc/e$. Hamiltonian (1) depends on A_{xy} through the phases U_{xy} and thus the relevant physical quantity is the product of the phases around a plaquette, i.e., $U(P) \equiv \exp[iC(P)]$. If $U(P)$ is not 1, $\mathcal{H}$ is frustrated. In this paper we assume $U(P)$ to be independent of the chosen plaquette, i.e., that

$$U(P) = e^{2\pi i f}, \tag{3}$$

with $0 \leqslant f \leqslant 1$, independent of P. Using the invariance of the Hamiltonian under the transformation $\psi_x \to \psi_x^*$, it is not restrictive to take f in the interval $0 \leqslant f \leqslant 1/2$. We will work in a finite lattice of size L^2 with periodic boundary conditions. Therefore, we have

$$\prod_P U(P) = 1, \tag{4}$$

where the product is extended over all lattice plaquettes. This implies that $f L^2$ must be an integer.

Hamiltonian (1) is invariant under the local gauge transformations

$$\psi_x \to V_x \psi_x, \qquad U_{xy} \to V_x^* U_{xy} V_y, \tag{5}$$

where V_x is a phase, $|V_x| = 1$. Physical observables must be gauge invariant. For such observables, the choice of the fields A_{xy} is irrelevant: only the value of f is relevant. In a finite volume, this statement is strictly true only if free boundary conditions are taken. If one considers periodic boundary conditions, one must also specify the value of $\exp\big(i \sum A_{xy}\big)$ along two non-trivial lattice paths that wind around the lattice (they are sometimes called Polyakov loops). For instance, one must also fix $P_1(m) = \exp\big(i \sum_n A_{(n,m),(n+1,m)}\big)$ and $P_2(m) = \exp\big(i \sum_n A_{(m,n),(m,n+1)}\big)$ for some fixed values of m. If we require the absence of magnetic circulation along these non-trivial paths, we must have $P_1(m) = P_2(m) = 1$ for any m. On a finite lattice of size L^2, this condition can be satisfied only if $f L$ is an integer, a condition that will always be satisfied in the numerical simulations that we shall present.

The critical behavior of uniformly frustrated XY models changes dramatically with f. For $f = 0$ the model corresponds to the standard XY model, which is not frustrated. It shows a Kosterlitz–Thouless transition at T_{KT} (on a square lattice [2] $T_{\mathrm{KT}} = 0.892\,94(8)$), where the correlation length ξ diverges as $\ln \xi \sim (T - T_{\mathrm{KT}})^{-1/2}$ for $T \gtrsim T_{\mathrm{KT}}$; the low-temperature phase, $T < T_{\mathrm{KT}}$, is characterized by quasi long-range order—correlation functions decay algebraically—associated with a line of fixed points. In the case of maximal frustration, i.e. for $f = 1/2$, the system undergoes two very close continuous transitions (their critical temperature is $T \approx 0.45$ on the square lattice), respectively in the Ising and Kosterlitz–Thouless universality classes, see e.g., [3, 4] and references therein. The critical behavior for other values of f is even more complex, see e.g., [5–13] for experiments. There may be several transitions, whose nature is not clear in most of the cases. Even the structure of the ground state is only partially understood [14–16]. For $f = 1/n$, where n is an integer number, if T_c is the critical temperature where the paramagnetic phase ends, T_c decreases with increasing n; for example, [9] $T_c \lesssim 0.22$ for $f = 1/3$ and [8] $T_c \lesssim 0.05, 0.03$ for $n = 30$ and 56, respectively. These studies suggest that T_c vanishes [7, 8] as $T_c \sim 1/n$ when $n \to \infty$. The critical behavior for irrational values of f is even less clear, see e.g., [11, 12]. In this case, there are some indications that the system is paramagnetic for any T and that a glassy transition occurs at zero temperature [12].

The above-mentioned works studied the critical behavior as a function of the temperature T, while keeping the uniform frustration f fixed. In this paper we investigate a different critical limit, i.e., we consider the limit $f \to 0$ at fixed T in the region $T \leqslant T_{\mathrm{KT}}$. In other words,

we investigate the effect of a small uniform frustration on the low-temperature XY critical behavior. We show that a uniform frustration is a relevant perturbation at the fixed points that occur in the XY model for $T \leqslant T_{KT}$. As soon as f is non-vanishing, the correlation length becomes finite and the system is paramagnetic.

The critical behavior for small values of f can be understood within the Coulomb-gas picture [17]. If one considers the Villain Hamiltonian corresponding to (1), one can write the partition function as

$$Z_{\text{Villain}} = \int \prod_x d\theta_x \, e^{-\beta \mathcal{H}} = Z_{\text{SW}} \sum_{\{n_x\}} \exp(2\pi\beta\mathcal{H}_{\text{CG}}), \qquad (6)$$

where [17] Z_{SW} is the spin-wave contribution and $\mathcal{H}_{\text{CG}}$ is the Coulomb-gas Hamiltonian:

$$\mathcal{H}_{\text{CG}} = \frac{1}{2} \sum_{ij} (n_i - f) V(r_i - r_j)(n_j - f), \qquad (7)$$

where n_i is an integer (vorticity) defined at the site i of the dual lattice and $V(r)$ is the lattice Coulomb potential. In (6) the sum over n_x is restricted to configurations satisfying the neutrality condition [17] $\sum_i (n_i - f) = 0$. For $f = 0$ and $T < T_{KT}$ this representation allows one to show that correlation functions decay algebraically. The two-point correlation function is the product of a spin-wave contribution, which decays algebraically, and of a vortex contribution. For $T < T_{KT}$ charged vortices are strictly bound to form dipoles and the corresponding correlation function also decays algebraically [18]. For $f > 0$ the picture changes. For small f, in the temperature interval $f T_{KT} < T < T_{KT}$, there are unbounded particles with $n = 0$ and charge $-f$, which screen the Coulomb interaction among the vortices of charge $n - f \approx n, n \neq 0$. The Debye screening length can be easily computed. Consider a vortex of charge 1, surrounded by particles of charge $-f$. Since there is one charge $-f$ for each lattice site, complete screening is achieved when these charges occupy a circle of area A, such that $Af = 1$. Thus, the screening length ξ should be proportional to $f^{-1/2}$. In this picture, for $f \to 0$, the system is equivalent to a dilute gas (the density is proportional to $f^{1/2}$) of neutral particles interacting by means of a screened Coulomb potential $V_{\text{sc}}(r)$. We can thus perform a standard virial expansion to predict that the vortex–vortex correlation function is proportional to $V_{\text{sc}}(r)$, hence decays exponentially with a rate controlled by the Debye screening length. This argument indicates that, for sufficiently small f and any $T < T_{KT}$, the system is paramagnetic with a correlation length that scales as

$$\xi \sim f^{-1/2}, \qquad (8)$$

for $f \to 0$. It is worth mentioning that also an analysis of the ground-state configurations shows the emergence of a typical length scale, associated with the ground-state modulation, which scales as $f^{-1/2}$ [19].

Equation (8) can also be predicted by simple dimensional arguments. For a given value of f and T, consider a real-space renormalization-group (RG) transformation. Eliminate lattice sites obtaining a lattice with a link length that is twice that of the original lattice. In lattice units we have $\xi' = \xi/2$, where we use a prime for quantities that refer to the decimated lattice. Analogously, we obtain $f' = 4f$ for the frustration parameter. It follows $\xi' f'^{1/2} = \xi f^{1/2}$. This quantity is therefore constant under RG transformations, i.e., $\xi f^{1/2} = c$. Under the RG transformation, the Hamiltonian parameters also change. In particular, the transformation induces a temperature change $T \to T'$. However, for small f, one is close to the XY line of fixed points and thus we expect $T' \approx T$. Thus, the condition $\xi f^{1/2} = c$ holds at (approximately) fixed temperature and $f \to 0$. Therefore, it implies (8).

In this paper, we wish to verify numerically (8) and study the critical behavior of gauge-invariant susceptibilities (they will be defined in the following section). Note that, in a sense,

at fixed $T \leqslant T_{\mathrm{KT}}$, the magnetic flux f plays the role of the reduced temperature, with an associated correlation length exponent $\nu = 1/2$.

The paper is organized as follows. In section 2 we define gauge-invariant correlation functions, the associated susceptibilities and correlation lengths, and discuss the expected critical behavior. In section 3 we present some Monte Carlo (MC) results that fully confirm the theoretical predictions.

2. Definitions and general scaling properties

In order to check prediction (8), we consider two different gauge-invariant correlation functions:

$$G_{\mathrm{sq}}(x; y) \equiv |\langle \psi_x \psi_y^* \rangle|^2, \qquad G_\Gamma(x; y) \equiv \langle \mathrm{Re}\, \psi_x U[\Gamma_{x;y}] \psi_y^* \rangle. \qquad (9)$$

Here $\Gamma_{x;y}$ is a path that connects sites x and y and $U[\Gamma_{x;y}]$ is a product of phases associated with the links that belong to $\Gamma_{x;y}$. More precisely, if a link $\langle wz \rangle$ belongs to the path, w and z have coordinates $w = (w_1, w_2)$ and $z = (z_1, z_2)$, such that $z_1 - w_1 \geqslant 0$ and $z_2 - w_2 \geqslant 0$, we define $R_{wz} = U_{wz}$ if point w occurs before point z while moving along the path; otherwise, we set $R_{wz} = U_{wz}^*$. The phase $U[\Gamma_{x;y}]$ is the product of all the phases R_{wz} associated with the links belonging to the path.

Definition (9) of $G_\Gamma(x; y)$ depends on a family of paths $\Gamma = \{\Gamma_{x;y}\}$. We assume this family to be translationally invariant: the path $\Gamma_{x;y}$ is obtained by rigidly translating the path $\Gamma_{0;y-x}$ that connects the origin to $y - x$. In this case, the correlation function $G_\Gamma(x; y)$ is uniquely defined by specifying the paths from the origin to any point x.

Because of the presence of the gauge field, the Hamiltonian is not translationally invariant, nor it is symmetric under the symmetry transformations of the lattice. Nonetheless, there are generalized symmetries of the Hamiltonian that also involve gauge transformations. For instance, if Lf is an integer, the Hamiltonian is invariant under the generalized translations

$$\psi'_{(n,m)} = \psi_{(n+1,m)} U_{(n,m),(n+1,m)}^* \, \mathrm{e}^{-2\pi \mathrm{i} m f},$$
$$\psi'_{(n,m)} = \psi_{(n,m+1)} U_{(n,m),(n,m+1)}^* \, \mathrm{e}^{2\pi \mathrm{i} n f}. \qquad (10)$$

Gauge-invariant correlation functions are invariant under these transformations. This implies that they do not depend on x and y separately, but only on the difference $y - x$. This invariance can be understood intuitively if one notes that gauge-invariant quantities should only depend on the value of the flux through a plaquette, i.e., $U(P)$, and of the Polyakov correlations $P_1(m)$ and $P_2(m)$. In our model $U(P)$ is independent of P and, if Lf is an integer, $P_1(m)$ and $P_2(m)$ do not depend on m: hence, a translation invariance holds.

Analogously, the Hamiltonian is invariant under generalized transformations that involve lattice symmetries and gauge transformations. For instance, in infinite volume the Hamiltonian is invariant under the generalized reflection transformations

$$\psi'_{(n,m)} = \psi_{(-n,m)}^* K_m^* \prod_{k=0}^{|n|-1} [U_{(k,m),(k+1,m)} U_{(-k-1,m),(-k,m)}^*], \qquad (11)$$

where

$$K_m = \begin{cases} \prod_{k=0}^{m-1} U_{(0,k),(0,k+1)}^2 & \text{for} \quad m \geqslant 1, \\ 1 & \text{for} \quad m = 0, \\ \prod_{k=0}^{-m-1} U_{(0,k+m),(0,k+m+1)}^{*2} & \text{for} \quad m \leqslant -1. \end{cases} \qquad (12)$$

Under these symmetries $G_{\mathrm{sq}}(x; y)$ transforms covariantly. If T is a lattice symmetry, $G_{\mathrm{sq}}(x; y) = G_{\mathrm{sq}}(Tx; Ty)$. These relations do not hold in general for $G_\Gamma(x; y)$ since a lattice symmetry also changes the path family.

Given $G_\Gamma(x; y)$ and $G_{sq}(x; y)$, we define the corresponding susceptibilities

$$\chi_\Gamma \equiv \sum_y G_\Gamma(x; y), \qquad \chi_{sq} \equiv \sum_y G_{sq}(x; y), \tag{13}$$

where sums are extended over all lattice points y. Because of the translational invariance, χ_{sq} and χ_Γ do not depend on the point x. Of course, χ_Γ depends on the family of paths $\Gamma = \{\Gamma_{x;y}\}$. Then, for any gauge-invariant correlation function $G(x; y)$ we define on a finite lattice of size L^2

$$F \equiv \sum_{y \equiv (y_1, y_2)} \cos[q_{min}(y_1 - x_1)] G(x; y) \tag{14}$$

where $x \equiv (x_1, x_2)$ and $q_{min} \equiv 2\pi/L$. The correlation length is defined by[4]

$$\xi^2 \equiv \frac{1}{4 \sin^2(q_{min}/2)} \frac{\chi - F}{F}. \tag{15}$$

Note that an equally good definition of F is

$$F \equiv \sum_{y \equiv (y_1, y_2)} \cos[q_{min}(y_2 - x_2)] G(x; y). \tag{16}$$

For the correlation function $G_{sq}(x; y)$, one can show that these two definitions of F are equivalent, but this is not generically the case of $G_\Gamma(x; y)$, since this quantity is not symmetric under lattice transformations. In the following we use definition (14) for F.

In the introduction we derived a prediction for the correlation length, $\xi \sim f^{-1/2}$. We wish now to obtain a similar result for the susceptibilities. In order to predict their scaling behavior, let us note that, for $f = 0$ and $T \leqslant T_{KT}$, $\langle \psi_0 \psi_x^* \rangle$ decays algebraically, i.e., $\langle \psi_0 \psi_x^* \rangle \sim x^{-\eta(T)}$. The critical exponent $\eta(T)$ depends on T and varies between $\eta(0) = 0$ and $\eta(T_{KT}) = 1/4$. For $f \neq 0$, it is natural to assume that

$$\begin{aligned}
\chi_\Gamma &\sim \int_{x<\xi} \mathrm{d}^2x \, x^{-\eta(T)} \sim \xi^{2-\eta(T)} \sim f^{-1+\eta(T)/2}, \\
\chi_{sq} &\sim \int_{x<\xi} \mathrm{d}^2x \, x^{-2\eta(T)} \sim \xi^{2-2\eta(T)} \sim f^{-1+\eta(T)}.
\end{aligned} \tag{17}$$

In particular, these equations predict $\chi_\Gamma \sim f^{-7/8}$ and $\chi_{sq} \sim f^{-3/4}$ at $T = T_{KT}$.

The check of the previous prediction for χ_{sq} does not present conceptual difficulties. Instead, when considering χ_Γ, one should keep in mind that this quantity depends on a path family. Thus, there is a natural question that should be considered first. Given a path family $\Gamma^{(f_1)}$ for a given value $f = f_1$ of the frustration parameter, we must specify which path family $\Gamma^{(f_2)}$ must be considered for $f = f_2 \neq f_1$. Only if $\Gamma^{(f_2)}$ is chosen appropriately, does the relation

$$\frac{\chi_{\Gamma^{(f_1)}}}{\chi_{\Gamma^{(f_2)}}} \approx \left(\frac{f_1}{f_2}\right)^{-1+\eta(T)/2} \tag{18}$$

hold for $f_1, f_2 \to 0$. A naive choice would be $\Gamma^{(f_1)} = \Gamma^{(f_2)}$. As we now discuss, this choice is not correct: different path families should be chosen for different values of f.

To clarify this issue, let us imagine we are working in the continuum. For each f, let us consider a family of paths $\Gamma^{(f)} = \{\Gamma_{x;y}^{(f)}\}$. Because of the translation invariance, we can limit ourselves to paths going from the origin to any point y. These paths can be parametrized in

---

[4] Note that this definition of ξ corresponds to $\xi^2 = \sum_y (y_1^2/2) G(0; y)/\chi$ in the infinite-volume limit.
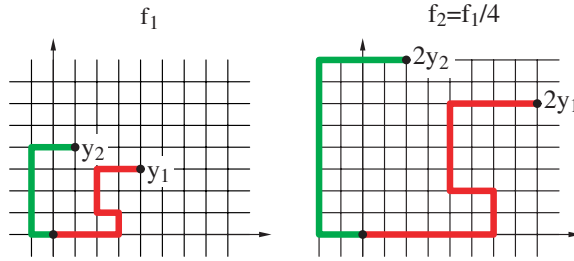
Figure 1. On the left, we report two paths connecting the origin to y_1 and y_2, respectively. On the right, we report the corresponding paths connecting the origin to $2y_1$ and $2y_2$. The figure on the left corresponds to a frustration parameter $f = f_1$, that on the right to $f = f_2 = f_1/4$.

terms of a function $X^{(f)}(t; y)$ such that $X^{(f)}(0; y) = 0$ for all y, $X^{(f)}(1; y) = y$. The path from the origin to y is given by

$$x = X^{(f)}(t; y) \qquad t \in [0, 1]. \tag{19}$$

To determine the relation between $\Gamma^{(f_1)}$ and $\Gamma^{(f_2)}$, one should remember that x/ξ should be kept fixed in the critical limit. Thus, we expect the path family to be invariant only if all lengths are expressed in terms of ξ. In other words, set $\bar{x} = x/\xi_f$, $\bar{y} = y/\xi_f$ and rewrite (19) as

$$\bar{x} = \frac{1}{\xi_f} X^{(f)}(t; \bar{y}\xi_f) \qquad t \in [0, 1], \tag{20}$$

where ξ_f is the correlation length for the system with frustration parameter f. The natural requirement is therefore that the right-hand side be independent of f, that is

$$\frac{1}{\xi_{f_2}} X^{(f_2)}(t; \bar{y}\xi_{f_2}) = \frac{1}{\xi_{f_1}} X^{(f_1)}(t; \bar{y}\xi_{f_1}). \tag{21}$$

Since we expect $\xi_f \sim f^{-1/2}$, we obtain the relation

$$X^{(f_2)}(t; ry) = r X^{(f_1)}(t; y), \qquad r = \left(\frac{f_1}{f_2}\right)^{1/2}. \tag{22}$$

In figure 1 we report an example corresponding to $f_1 = 4f_2$. The paths from the origin to y_1 and y_2 which belong to $\Gamma^{(f_1)}$ completely fix the paths to $2y_1$ and $2y_2$ belonging to $\Gamma^{(f_2)}$. Of course, on the lattice it is impossible to ensure (22) exactly. However, note that the relevant scale is fixed by the correlation length and thus, violations at the level of the lattice spacing are irrelevant in the critical limit.

In the following we shall consider the path families $\Gamma_n \equiv \{\Gamma_{n;0;x}\}$, which are specified by a non-negative integer n. They are defined as follows (see figure 2). The path $\Gamma_{n;0;x}$ connecting the origin to the point $x \equiv (x_1, x_2)$ consists of three segments: the first one connects the origin to $(-n, 0)$; the second one goes from $(-n, 0)$ to $(-n, x_2)$; the last one is horizontal, from $(-n, x_2)$ to point x. We indicate with $\chi_n(f)$ the corresponding susceptibilities and with $\xi_n(f)$ the corresponding correlation lengths. These families of paths behave simply under transformation (22). If we consider the path $\Gamma_{n;0;x}$ for $f = f_1$, mapping (22) implies that, for $f = f_2$, one should consider the path $\Gamma_{rn;0;rx}$ between the origin and the point rx. This implies that, if we take the path family Γ_n for $f = f_1$, we must consider Γ_{rn} for $f = f_2$. As a consequence, χ_n and ξ_n scale correctly only if we consider the limit $n \to \infty$, $f \to 0$ at fixed $nf^{1/2}$. Thus, we predict the scaling behaviors

$$\chi_n = f^{-1+\eta(T)/2} F_\chi(nf^{1/2}), \qquad \xi_n = f^{-1/2} F_\xi(nf^{1/2}), \tag{23}$$
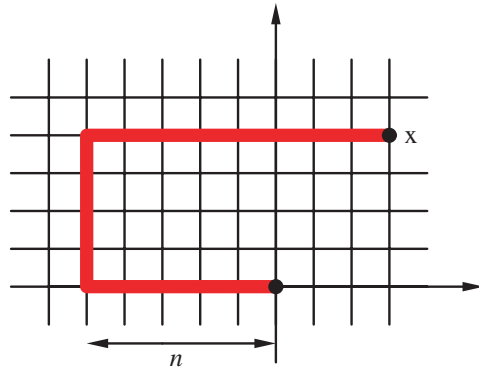
Figure 2. The path connecting the origin to the point x which belongs to the path family Γ_n.

where $F_\chi(x)$ and $F_\xi(x)$ are appropriate scaling functions. In the following section, we verify these predictions.

3. Numerical results

We perform simulations for various values of $f = 1/m$, m integer and T in the interval $T \leqslant T_{KT}$, where T_{KT} is the critical temperature of the XY model, $T_{KT} = 0.89294(8)$ [2]. We consider finite lattices of size L^2, where L is a multiple of $1/f$, and periodic boundary conditions for the spins. Since we perform MC simulations in a gapped phase, boundary conditions are expected to be irrelevant in the thermodynamic limit. Cluster algorithms cannot be used in the presence of frustration and thus we use an overrelaxed algorithm, which consists in performing microcanonical and Metropolis updates. Predictions (8) and (17) hold in the thermodynamic limit, i.e., for sufficiently large values of the ratio L/ξ, where finite-size effects are negligible. We find numerically that size effects are much smaller than our statistical errors for $Lf \gtrsim 3$.

In the simulations we choose the gauge

$$
\begin{aligned}
A_{xy} &= 0 \quad \text{if} \quad y = x + \hat{1}, \\
A_{xy} &= 2\pi f x_1 \quad \text{if} \quad y = x + \hat{2},
\end{aligned}
\tag{24}
$$

which is consistent with (3) and $P_1(m) = P_2(m) = 1$, as long as L is an integer multiple of $1/f$. With this gauge choice the computation of the susceptibilities χ_n and of the corresponding correlation lengths ξ_n is quite simple. Indeed, $U[\Gamma_{n;x;y}] = 1$ for any y if the first component of x is n, i.e., if $x = (n, m)$, m is arbitrary. Thus, if we choose $x = (n, m)$ in definition (13), we can compute χ_n without taking into account the phases U_{xy}. In practice, we have determined χ_n by using
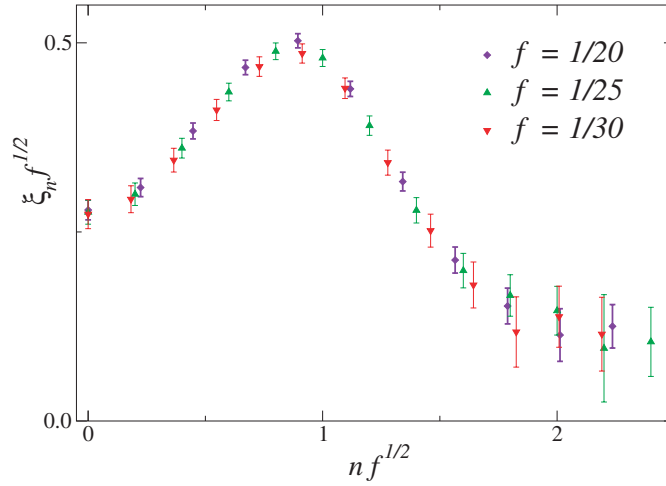
$$
\chi_n = \frac{1}{L} \sum_m \sum_y \langle \mathrm{Re}\, \psi_{(n,m)} \psi_y^* \rangle.
\tag{25}
$$

An analogous expression holds for the correlation lengths.

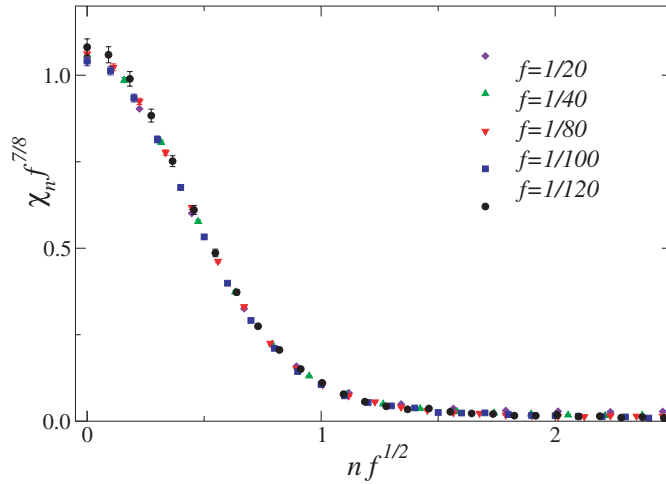In figures 3 and 4 we plot the correlation lengths ξ_n and the susceptibilities χ_n at $T = T_{KT}$ for several values of f and n. In this case $\eta(T) = 1/4$ so that χ_n should scale as $f^{-7/8}$. It is easy to show that

$$
\chi_n = \chi_{n+1/f}, \qquad \chi_n = \chi_{1/f-n},
\tag{26}
$$

Figure 3. Scaling plot for the correlation length ξ_n at T_{KT}. For each f we report the data satisfying $n \leqslant 1/(2f)$.

Figure 4. Scaling plot for the susceptibilities χ_n at T_{KT}. For each f we report the data satisfying $n \leqslant 1/(2f)$.

so that in (23) one must restrict oneself to data satisfying $0 \leqslant n \leqslant 1/(2f)$. The results reported in the figures show the scaling behavior (23) quite precisely, confirming the theoretical arguments. Note that the scaling function $F_\chi(x)$ apparently goes to zero as x increases. This behavior will be confirmed below by the analysis of a non-gauge-invariant correlation function.

Good agreement is also found at $T < T_{KT}$. We check the behavior of $\chi_{n=0}$ (in this case, the same path family can be used for all values of f) up to $T = 0.2$. At $T = 0.2, 0.3, 0.4, 0.5, 0.8$, a fit of χ_0 to $af^{-1+\eta(T)/2}$ gives $\eta = 0.042(8), 0.050(6), 0.079(6), 0.098(7), 0.171(3)$. These results are in substantial agreement with the leading spin-wave contribution $\eta = T/(2\pi)$, and the MC estimates [20] $\eta = 0.036(3), 0.052(5), 0.074(6), 0.100(8), 0.19(2)$. For example, in figure 5 we show the MC results for χ_0 at $T = 0.4$, together with the result of the fit. The data show a clear power-law behavior in perfect agreement with (17).
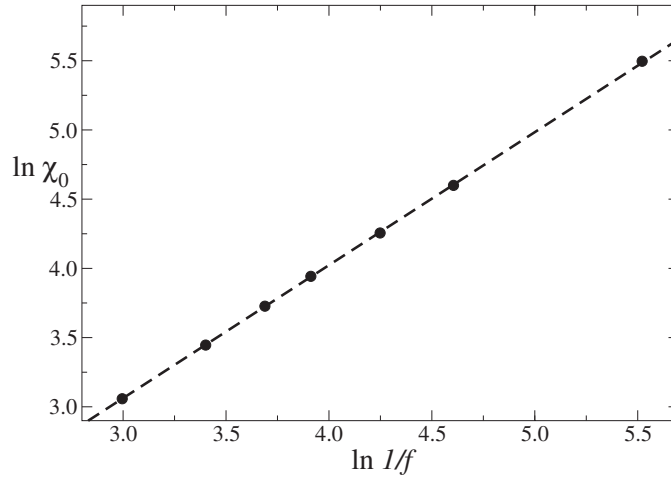
Figure 5. Critical behavior of χ_0 versus $1/f$ at $T = 0.4$. The line is the results of a fit to $\chi = af^{-\varepsilon}$, which gives $\varepsilon = 0.961(3)$, corresponding to $\eta(T = 0.4) = 0.079(6)$.

We also investigated the critical behavior of $\chi_{\rm sq}$, which is expected to scale as $f^{-3/4}$. For $1/f = 40, 60, 80$, we obtain $\chi_{\rm sq} = 9.933(7), 13.630(23), 17.06(4)$, respectively. These results are fully consistent with the theoretical prediction. Indeed, the product $f^{3/4}\chi_{\rm sq}$ clearly converges to a constant as $f \to 0$ (corrections are expected to be proportional to $1/\ln(1/f)$, as in the XY model at $T_{\rm KT}$): we have $f^{3/4}\chi_{\rm sq} = 0.6245(5), 0.6322(11), 0.6378(15)$ for the same values of f.

Finally, we mention that correlation functions which are not gauge invariant show a different behavior. For example, one may consider the susceptibility χ_w associated with the two-point function $\langle {\rm Re}\, \psi_x \psi_y^* \rangle$ in the gauge (25):

$$\chi_w = \frac{1}{L^2} \sum_{x,y} \langle {\rm Re}\, \psi_x \psi_y^* \rangle. \tag{27}$$

At $T_{\rm KT}$ it shows a power-law behavior $\chi_w \sim f^{-\varepsilon}$ as well, but with a power $\varepsilon \approx 0.39$, definitely different from the value 0.875 of the gauge-invariant definition. This result can be derived analytically. Indeed, we can rewrite
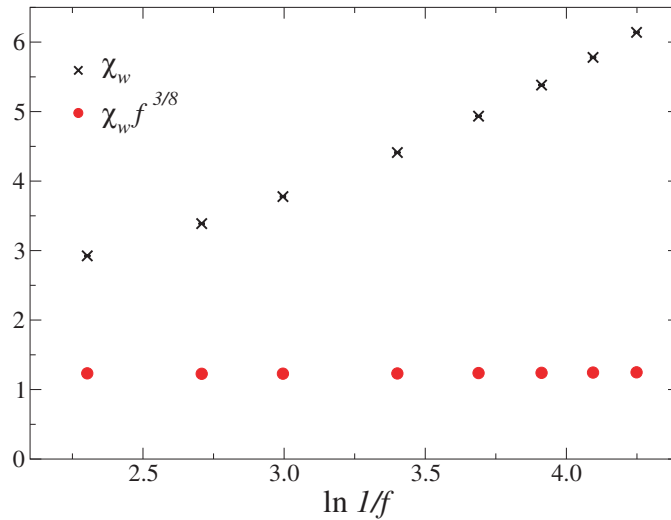
$$\chi_w = \frac{1}{L} \sum_{n=0}^{L-1} \chi_n, \tag{28}$$

where χ_n is defined in (25). Using the properties (26) of the susceptibilities χ_n, (28) can be rewritten as

$$\chi_w \approx 2f \sum_{n=0}^{1/(2f)} \chi_n. \tag{29}$$

In this range of values of n, as is clear from figure 4, we can use the scaling behavior (23) and write

$$
\begin{aligned}
\chi_w &\sim f \times f^{-7/8} \int_0^{1/(2f)} {\rm d}n\, F(nf^{1/2}) \\
&\sim f^{-3/8} \int_0^{1/(2f^{1/2})} {\rm d}x\, F(x) \sim f^{-3/8} \int_0^\infty {\rm d}x\, F(x).
\end{aligned} \tag{30}
$$

Figure 6. MC results for the non-gauge-invariant susceptibility χ_w and for the product $f^{3/8}\chi_w$ versus $\ln 1/f$ at $T = T_{KT}$.

Thus, provided that $F(x)$ is integrable (we already noted that the MC data for χ_n are consistent with $F(x) \to 0$ as $x \to \infty$), we predict $\chi_w \sim f^{-3/8} = f^{-0.375}$, which is consistent with the MC data (see figure 6).

Note that the critical behavior of χ_w depends on the chosen gauge. If we use the gauge

$$
\begin{aligned}
A_{xy} &= -\pi f x_2 \quad \text{if} \quad y = x + \hat{1}, \\
A_{xy} &= \pi f x_1 \quad \text{if} \quad y = x + \hat{2},
\end{aligned}
\tag{31}
$$

the susceptibility χ_w does not diverge and approaches a constant as $f \to 0$.

In conclusion, we have shown that a small amount of uniform frustration (equivalently, a small uniform magnetic field) destabilizes the line of fixed points that occur in the XY model for $T \leqslant T_{KT}$. As soon as f is different from zero, the system becomes paramagnetic. The critical behavior $\xi \sim f^{-1/2}$ can be predicted by simple Coulomb-gas and scaling arguments. Our numerical simulations fully confirm this prediction. Also the scaling behavior (17) for the magnetic susceptibilities is fully consistent with the numerical results.

References

[1] Fazio R and van der Zant H 2001 *Phys. Rep.* **355** 235
[2] Hasenbusch M 2005 *J. Phys. A: Math. Gen.* **38** 5869
     Hasenbusch M and Pinn K 1997 *J. Phys. A: Math. Gen.* **30** 63
[3] Hasenbusch M, Pelissetto A and Vicari E 2005 *J. Stat. Mech.: Theory Exp.* P12002
     Hasenbusch M, Pelissetto A and Vicari E 2005 *Phys. Rev.* B **72** 184502
[4] Korshunov S E 2006 *Usp. Fiz. Nauk* **176** 233
     Korshunov S E 2006 *Phys.Usp.* **49** 225 (Engl. Transl.)
[5] Choi M Y and Doniach S 1985 *Phys. Rev.* B **31** 4516
[6] Korshunov S E, Vallat A and Beck H 1995 *Phys. Rev.* B **51** 3071
[7] Franz M and Teitel S 1995 *Phys. Rev.* B **51** 6551
[8] Hattel S A and Wheatley J M 1995 *Phys. Rev.* B **51** 11951
[9] Lee S and Lee K-C 1995 *Phys. Rev.* B **52** 6706
[10] Straley J P, Morozov A Y and Kolomeisky E B 1997 *Phys. Rev. Lett.* **79** 2534

[11] Choi M Y and Stroud D 1985 *Phys. Rev.* B **32** 7532
     Choi M Y and Stroud D 1987 *Phys. Rev.* B **35** 7109
[12] Park S Y, Choi M Y, Kim B J, Jeon G S and Chung J S 2000 *Phys. Rev. Lett.* **85** 3484
[13] Ling X S *et al* 1996 *Phys. Rev. Lett.* **76** 2989
     Ling X S *et al* 1996 *Phys. Rev. Lett.* **77** 410 (errata)
[14] Teitel S and Jayaprakash C 1983 *Phys. Rev. Lett.* **51** 1999
[15] Straley J P and Barnett G M 1993 *Phys. Rev.* B **48** 3309
[16] Lee S J, Lee J R and Kim B 2002 *Phys. Rev. Lett.* **88** 025701
[17] Fradkin E, Huberman B A and Shenker S H 1978 *Phys. Rev.* B **18** 4789
[18] José J V, Kadanoff L P, Kirkpatrick S and Nelson D R 1977 *Phys. Rev.* B **16** 1217
[19] Nussinov Z 2001 *Preprint* cond-mat/0107339
[20] Berche B 2003 *J. Phys. A: Math. Gen.* **36** 586